How cardinal utility entered economic analysis, 1909-1944

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Abstract

The paper illustrates the methodological and analytical issues that characterized, as well as the personal and institutional aspects that informed the discussions leading to the definition of the current notion of cardinal utility as utility unique up to positive linear transformations. As originally this type of utility was not called “cardinal”, the paper also investigates the terminological question of when and how the expression “cardinal” was coupled with positive linear transformations. In opposition to existing narratives, the paper shows that cardinal utility entered economic analysis between 1909 and 1944, that is, during the ordinal revolution in utility theory.

Keywords

Cardinal utility; Ranking of transitions; Differences of utility; Ordinal revolution; Paul Samuelson.

JEL Classification

B13 (History of Economic Thought through 1925, Neoclassical);

B21 (History of Economic Thought since 1925, Microeconomics);

B40 (Economic Methodology, General);

D11 (Consumer Economics, Theory).

1. Introduction

In current economic theory, a utility function representing the preferences of an individual is called 'cardinal' if it is unique up to a positive linear transformation (see e.g. Fishburn 1970, 1987). This means that, if the utility function $U(x)$ represents the individual's preferences, another utility function obtained by multiplying $U(x)$ by a positive number α and then adding any number β, that is, a transformation of $U(x)$ having the form $\alpha U(x)+\beta$, with $\alpha>0$, also represents the individual's preferences.[1] The different assumptions generating cardinal utility in this specific sense and their economic interpretation will be discussed at length in the paper. Here it suffices to say that cardinal utility plays a prominent role in a number of areas of current microeconomics, such as the theory of decisions under risk, game theory, the theory of intertemporal decisions, and welfare analysis.[2] Other branches of microeconomics, such as demand analysis and general equilibrium theory, are instead based on ordinal utility, which is less restrictive. A utility function, in fact, is called 'ordinal' if it is unique up to *any*, and thus possibly non-linear, positive transformation. Formally, an ordinal utility function is unique up to transformations of the form $F[U(x)]$, where F is a function whose first derivative F' is positive.

The present paper reconstructs the progressive definition and stabilization of the notion of cardinal utility in the specific sense it has assumed in economic theory, that is, as utility unique up to positive linear transformations. The issue concerning the emergence of cardinal utility has a terminological counterpart. As originally utility unique up to positive linear transformations was not called 'cardinal', the paper also investigates the terminological question of when and how economists began to label as 'cardinal' the utility functions that are invariant to transformations of the form $\alpha U(x)+\beta$.

Despite the importance of the notion of cardinal utility in current microeconomic theory, the existing literature on utility theory and its history does not provide a satisfactory reconstruction of the origins of that notion. With some simplification, one may say that there exist two main narratives about the appearance of cardinal utility in economic analysis.

According to the first, cardinal utility entered economics during the so-called marginal revolution of the 1870s. In this reconstruction, William Stanley Jevons and the other early marginalists of the late nineteenth century were cardinalists; then, in the so-called ordinal revolution inaugurated by Vilfredo Pareto ([1909] 1971) and virtually concluded by John Hicks's *Value and Capital* (1939), utility theorists moved away from cardinalism and embraced an ordinal approach to utility (see e.g. Blaug 1987, Niehans 1990, Mandler 1999). According to the second narrative (see e.g. Harsanyi 1977, Binmore 2009), cardinal utility entered economics only after the completion of the ordinal revolution, and drew from the Expected Utility Theory that John von Neumann and Oskar Morgenstern put forward in their book *Theory of Games and Economic Behavior* (1944). The axioms of Expected Utility Theory imply in fact the existence of a utility function that can be interpreted as expressing the individual's preferences over the outcomes of risky alternatives, and that is unique up to positive linear transformations.

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[1] In the economic literature, the term 'linear' is often replaced by the technically more precise expression 'affine', so that transformations of the form $\alpha U(x)+\beta$, where $\alpha>0$, are called 'positive affine transformations'. In this paper I stick to the less technical and more intuitive term 'linear'.

[2] To appreciate the importance of cardinal utility in current microeconomics, see Mas-Colell, Whinston and Green (1995, chapters 6, 8, 20, and 22).

I contend that both narratives are flawed, and that cardinal utility entered economic analysis neither before the beginning of the ordinal revolution, nor after its completion, but *during the ordinal revolution*.

In opposition to the first narrative, I have demonstrated in a companion paper (Moscati 2013) that Jevons and the other founders of marginal utility were not cardinalists in the current sense of the term. They were not interested in positive linear transformations of the utility function, or in the ranking of utility differences that, as we will see in a moment, can be associated with these transformations. Rather, the early marginalists were interested in something more demanding, namely in measuring utility, which for them consisted in the possibility of expressing the utility of a commodity as a multiple of the utility of another commodity taken as unit.[3] But the second narrative is also misleading. As I show in the present paper, cardinal utility as utility unique up to positive linear transformations entered economic analysis well before the completion of the ordinal revolution and the appearance of von Neumann and Morgenstern's Expected Utility Theory. Moreover, the two authors of *Theory of Games* were perfectly aware of the prior debates concerning the assumptions implying cardinal utility.

More precisely, in this work I contend that cardinal utility was the outcome of a long-lasting discussion, inaugurated by Pareto himself in his *Manual of Political Economy* ([1909] 1971), as to an individual's capacity to rank transitions among different combinations of goods. This discussion continued through the 1920s and early 1930s and underwent a decisive acceleration between 1934 and 1938, that is, during the conclusive phase of the ordinal revolution. In particular, in 1934 Oskar Lange connected the ranking of transitions with utility unique up to positive linear transformations; in 1936 Franz Alt showed under what exact assumptions Lange's connection is valid; and in 1938 Paul Samuelson coupled the expression 'cardinal utility', which had been previously used with other meanings, with utility unique up to positive linear transformations. Between 1938 and 1944, that is in the heydays of ordinalism, this specific meaning of cardinal utility somehow stabilized yet the notion of cardinal utility remained of only marginal importance. The appearance of Expected Utility Theory in 1944 gave cardinal utility a justification apparently more convincing than the one based on the ranking of transitions, and propelled its use in economics. By showing how cardinal utility entered and stabilized in economic analysis during the ordinal revolution, the paper not only revises the conventional narratives of the history of cardinal utility, but also adds to a series of recent studies that have enriched and partially modified the standard picture of the ordinal revolution itself (Lenfant 2006, 2012; Hands 2010, 2011).

The history of the progressive definition and stabilization of the notion of cardinal utility is complex for various reasons. In the first place, it is multi-character play with a fairly long list of *dramatis personæ*. Besides Pareto, Lange, Alt and Samuelson, many other economists, such as John Hicks, Roy Allen, Henry Phelps Brown, Harro Bernardelli, and Joseph Schumpeter, contributed to the emergence of cardinal utility. Also Morgenstern, even before his encounter with von Neumann in Princeton in 1938, played a role in the events. Secondly, the action played out over a long time period, namely from 1909, when Pareto began the

[3] Measuring utility based on a unit is more demanding than ranking utility differences because the existence of a utility unit allows the ranking of utility differences, while the ranking of utility differences does not permit the identification of a utility unit and thereby to express another utility as a multiple of that unit. I discuss the relationship between unit-based measurement of utility and ranking utility differences in greater detail in Moscati (2013).

discussion about the ranking of transitions among different combinations of goods, to 1944, when Expected Utility Theory modified the interpretation of cardinal utility that had emerged in the 1930s.

Finally, the history of the consolidation of the notion of cardinal utility is complex because it includes various distinct dimensions, which in the paper I illustrate and connect. In the first place, the discussions about cardinal utility contained a major analytical element: economists long identified the ranking of transitions with the ranking of utility differences, but when this identification was challenged the problem arose of defining the exact assumptions under which the utility function is unique only up to linear transformations. Alt's 1936 solution to this problem represented one of the early applications of Hilbert's axiomatic method to economics.

An issue of priority further complicates the analysis of the role of Alt. Samuelson in 1938 also stated conditions that restrict the admissible transformations of the utility function to the positive linear ones; however Samuelson did not cite Alt. Based on archival research into Samuelson's papers at Duke University, I show that Samuelson was aware of Alt's contribution.

Furthermore, the history of cardinal utility exhibits significant epistemological aspects. In the first place, different stances on the ranking of transitions reflected the different methodological views regarding demand analysis that confronted each other during the ordinal revolution. At one extreme, some economists saw the ranking of transitions as a scientifically legitimate way of preserving decreasing marginal utility and other intuitive but non-ordinal notions of pre-Paretian demand analysis. At the other extreme were those who criticized that ranking as empirically unverifiable and superfluous to demand theory. Furthermore, the debate on the implications of transition ranking helped economists to understand the differences between the properties of preferences and the properties of the numbers used to represent preferences.

The paper also brings to light the personal and institutional dimensions of the discussions leading to the definition of cardinal utility. Most of the economists involved knew each other, often through their participation in the meetings of the Econometric Society, and were located in a limited number of institutions, primarily the London School of Economics (LSE), the University of Vienna and Harvard University. Since Alt played an important role but is little known in the history of economics, the paper also investigates the biographical background that led him to contribute to the definition of cardinal utility.

Finally, the paper calls attention, apparently for the first time, to the pivotal role of Samuelson in defining and popularizing the current meaning of cardinal utility. In so doing, the paper also modifies a widespread image of the American economist according to which his chief goal during the ordinal revolution was to free economic theory from any vestige of the utility concept.

2. Pareto and the ranking of transitions

In his *Manual*, Pareto ([1909] 1971: 112 and 396) maintained that utility cannot be measured, i.e. that it is impossible to identify a unit of utility and express the utility of commodities as a multiple of that unit. More importantly, Pareto showed that the main results of demand and equilibrium analysis are in fact independent of the measurability of utility, and can be drawn from the single assumption that individuals are able to rank combinations of goods. Under

this assumption, utility can be expressed by indices that represent the preference ranking of the individual in the sense that, if the individual prefers one combination of goods over another, the former must have a larger index. In the Mathematical Appendix, Pareto connected utility indices with positive transformations, i.e., if U is a utility index representing the individual's preferences, also any transformation $F(U)$ with a positive first derivative F' represents them. This is the uniqueness up to positive transformations that, in current economic theory, still characterizes ordinal utility.

A theory based on ordinal utility indices implies the dismissal of earlier notions of utility theory that are not unique up to positive transformations. Among these notions are the principle of decreasing marginal utility and the definition of complementary and substitute goods based on how the marginal utility of one good varies when the quantity of another varies.[4] It is not clear whether Pareto was fully aware of these restrictive implications of the ordinal approach. At any rate, when he needed decreasing marginal utility or the traditional definition of complementarity for considerations going beyond pure equilibrium analysis, he did refer to these notions (Bruni and Guala 2001).

Although Pareto considered utility unmeasurable, in one of the many digressions that embellish his *Manual* he cursorily suggested that we can imagine one special case in which utility can be measured. This happens when individuals: (1) are not only able to rank consumption alternatives, but (2) are also capable of ranking transitions from one alternative to another, and (3) are even capable of stating that a given transition is equally or twice as preferable as another.[5] For Pareto, assumption 2 was in accord with the idea of decreasing marginal utility and appeared plausible, at least for adjacent transitions. In particular, he claimed that this assumption restricts the arbitrariness of the utility index to those increasing transformations which display the following additional property:

> If in passing from [combination] I to [combination] II the man experiences more pleasure than in passing from II to III, the difference between the indices of I and II is greater than the difference between the indices of II and III. (192)

However, Pareto did not provide a mathematical characterization of how this property restricts the set of admissible transformations of utility indices. In particular, he did not associate assumption 2 with positive linear transformations of the indices. It should also be noted that, in the passage quoted above, Pareto takes for granted that the ranking of transitions from one combination to another implies the ranking of the differences between the utility indices associated with the combinations. As became clear much later, this implicit supposition is unwarranted.

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4 Let $U(x_i, x_j)$ be the utility function, and denote $U_{ij}=\partial^2 U/\partial x_i \partial x_j$. The principle of decreasing marginal utility implies that $U_{ii}<0$. However, the sign of U_{ii} is not invariant to increasing transformations of U. According to the definition used by early marginalists, two goods i and j are complementary if $U_{ij}>0$, and substitute if $U_{ij}<0$. However, not even the sign of U_{ij} is invariant to increasing transformations of U.

5 In the Mathematical Appendix, Pareto ([1909] 1971: 395–6) briefly considered another assumption under which utility could be measured, namely when the marginal utility of each commodity depends only on the quantity of that commodity. However, Pareto quickly discarded this hypothesis as farfetched, and so did most utility theorists after 1910. Thus the case of independent marginal utilities had no influence on the discussions that led to the definition of cardinal utility, and therefore is not relevant for the present paper. For more on Pareto's discussion of independent marginal utilities, see Montesano (2006).

With respect to assumption 3, Pareto discarded it as highly unrealistic and so, in the end, for him utility remained unmeasurable.

3. The ranking of transitions in the 1920s and early 1930s

In the 1920s and early 1930s, Pareto's discussion about the ranking of transitions and utility differences was picked up by a number of eminent economists from different quarters.[6] In Italy, the Paretian Luigi Amoroso endorsed the idea that individuals are capable of ranking transitions from one combination to another (i.e. Pareto's claim 2), and argued that this capacity makes the comparison of utility differences meaningful (1921: 91–2). This, in turn, allows preservation of the principle of decreasing marginal utility and the traditional definition of complementarity without returning to the pre-Paretian view that utility is measurable. In England, the LSE based economist and statistician Arthur Bowley stressed in his *The Mathematical Groundwork of Economics* that the principle of decreasing marginal utility and the traditional definition of complementarity require that individuals are able to rank transitions from one combination of goods to another (1924: 1–2). Bowley did not oppose this assumption. The Austrian economist Paul Rosenstein-Rodan admitted the possibility that individuals are able to rank utility differences but, like Pareto, denied that individuals are capable of stating 'how much larger or smaller the utility difference is' ([1927] 1960: 75).

Even the young Morgenstern touched upon the ranking of transitions and utility differences. Morgenstern (1902–1977) had studied under Hans Mayer, who then occupied one of the few chairs in economics at the University of Vienna. After completing his degree in economics in 1925, Morgenstern left Vienna on a three-year Rockefeller fellowship that allowed him, among other things, to visit LSE and Columbia University in New York. On his return in 1928, Morgenstern entered the University of Vienna as *Privatdozent* and joined Rosentein-Rodan as managing editor of the *Zeitschrift für Nationalökonomie*, a new economic journal that was edited by Mayer and enjoyed in the 1930s a significant international standing (Leonard 2010, Rothschild 2004). In 1931, in the German journal *Schriften des Vereins für Sozialpolitik*, Morgenstern published an article on 'Die drei Grundtypen der Theorie des subjektiven Wertes' (The three fundamental types of the theory of subjective value). Here he outlined the Austrian version of utility theory and presented the Lausanne and the Anglo-American versions of the theory as imperfect variations of the Austrian doctrine. Morgenstern also argued that economic subjects are able to compare not only utilities but also utility differences, and that these two abilities are all subjects need to behave rationally in the economy (1931: 13–14).

Two brief comments on the discussion of transition rankings in the 1920s and early 1930s are in order. To begin with, none of the economists contributing to the discussion employed the expression 'cardinal utility'. Secondly, although these economists often viewed the capacity of ranking transitions as a way of preserving some important notions of pre-Paretian utility theory within the boundaries of Pareto's ordinal framework, the exact meaning and implications of that capacity remained unexplored.

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[6] We mention here those who were mentioned in the debate of the mid-1930s. Among the others who touched upon the ranking of transitions are Alfonso de Pietri-Tonelli (1927) and Ragnar Frisch ([1926] 1971).

The discussion of the exact implications of transition ranking underwent a sudden and decisive acceleration in 1934. In that year, two junior members of the group of economists who had formed around Lionel Robbins after his 1929 appointment to the LSE chair in economics published a joint article that began the conclusive phase of the ordinal revolution initiated by Pareto.[7]

4. Cardinal and ordinal utility by Hicks and Allen

John Hicks (1904–1989) studied at Oxford, and joined the LSE in 1926 as a lecturer in economics, whereupon he commenced an intensive study of Pareto, Walras and Edgeworth. When Robbins arrived at the LSE in 1929, Hicks immediately became an enthusiast member of the Robbins circle and seminar. Like most of the protagonists of our story, Hicks was an early member of the Econometric Society, which had been founded in 1930 by Irving Fisher, Ragnar Frisch, and other prominent economists, and whose European meetings provided an important background to the events described in the present narrative. Roy Allen (1906–1983) studied mathematics at Cambridge and entered the LSE department of statistics as Bowley's assistant in 1928. He was another early affiliate of the Robbins group and early member of the Econometric Society.

In 1933, Hicks and Allen co-wrote 'A Reconsideration of the Theory of Value', which, after thorough discussion in Robbins' seminar (Hicks 1981), was published the following year in *Economica*, the senior LSE economics journal. In this article, Hicks and Allen (1934) endorsed Pareto's superseding of measurable utility, but argued that the Italian economist had not examined thoroughly what adjustments in demand analysis are made necessary by that superseding. They pointed out that some of the concepts Pareto used, such as the principle of decreasing marginal utility, are inconsistent with the immeasurability of utility and must therefore be eliminated. In effect, Hicks and Allen eliminated not only decreasing marginal utility, but also marginal utility, and even utility itself, and attempted to construct demand theory solely on the basis of observable choice behavior as captured by indifference curves. The cornerstone of their behaviorist analysis of demand became the marginal rate of substitution, which was understood as a quantitative and observable entity independent of utility.[8]

Hicks and Allen's article had an immediate and strong impact on economists working on demand analysis and became a standard reference for subsequent discussions on utility theory. For our story, it is relevant especially because it contributed in a significant way to the diffusion of the cardinal-ordinal terminology in economics.

As illustrated by Schmidt and Weber (2008) and Moscati (2013), the distinction between cardinal and ordinal numbers had been introduced in the late nineteenth century by German mathematicians, and had passed into economics through an article authored by the German mathematician and economist Andreas Voigt ([1893] 2008). Among the very few who took notice of Voigt's paper was Francis Ysidro Edgeworth (1894, 1900, 1907, 1915), who cursorily referred to Voigt's distinction between cardinal and ordinal numbers in four articles in the *Economic Journal*. It is important to stress that for Voigt and Edgeworth cardinal

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[7] On Robbins and his circle, see Howson (2011).

[8] More on Hicks and Allen's behaviorist approach to demand analysis in Fernandez-Grela (2006) and Moscati (2007).

numbers were completely unrelated to the ranking of differences between objects or to positive linear transformations of mathematical functions. Rather, for them cardinal numbers express the total number of units constituting a given quantity (for example, five), while ordinal numbers express the position of a specific unit of the quantity (for instance, the fifth unit). At any rate, before 1934 no other economist apart from Edgeworth seems to have employed the cardinal-ordinal terminology.

This situation changed when Hicks and Allen used the terms 'cardinal' and 'ordinal' in their influential 1934 article. As argued by Schmidt and Weber (2012), these two English economists had come across the cardinal-ordinal terminology either by reading Edgeworth or through Rosenstein-Rodan, who knew Voigt's article, had left Vienna for London in 1930, and subsequently had become an active member of the Robbins group. In one passage of their article Hicks and Allen (1934: 54–5) referred to Pareto's approach to utility theory as 'the "ordinal" conception of utility', and labeled the approaches relying on the measurability of utility as 'dependent upon a "cardinal" conception of utility'. While it is evident that by ordinal utility Hicks and Allen referred to Paretian utility indices that are unique up to positive transformations, they did not make clear what they meant by cardinal utility. Apparently, they used the notion of cardinal as a residual one, in the sense that they considered cardinal everything that is not ordinal, that is, not invariant to positive transformations of the utility function. Certainly, Hicks and Allen did not associate cardinal utility with the ranking of utility differences or positive linear transformations of the utility function.

Despite the fact that the terms 'ordinal' and 'cardinal' occurred only once in their paper, Hicks and Allen contributed immediately to their diffusion, at least within the Robbins group. Two other members of the group, namely Friedrich von Hayek and Frederic Benham, published separately in the November 1934 issue of *Economica* articles in which the terms are used (once in each). However, neither Hayek nor Benham associated the term cardinal with the ranking of utility differences.[9]

5. Lange and the determinateness of the utility function

The meaning and implications of the utility-difference ranking, as well as its relationships with the behaviorist and ordinalist approaches to demand analysis, were thoroughly investigated in a debate that took place after 1934, mainly in the *Review of Economic Studies*. The *Review* was the junior LSE economics journal, founded in 1933 by Ursula Webb, who belonged to the Robbins circle and in 1935 married Hicks; Abba Lerner, another brilliant member of the Robbins group; and Paul Sweezy, a Harvard graduate student who had visited the LSE in the academic year 1932–33. The debate was initiated by Lange, who was prompted to reconsider the discussions on transition rankings by Hicks's and Allen's claim that ordinal utility implies the abandonment of decreasing marginal utility and the traditional definition of complementary goods. Lange's article, published in the June 1934 issue of the *Review*, was entitled 'The determinateness of the utility function' (Lange 1934a). Since the

[9] Hayek (1934: 401) claimed that Carl Menger understood the numbers he used to express the marginal utility of goods 'not as cardinal but as ordinal figures'. Benham (1934: 446) argued that utility and welfare preclude 'objective measurement (whether in "cardinal" or "ordinal" numbers)'.

titles of many of the contributions generated by Lange's article repeated its title, the discussion he began can be labeled 'the debate on the determinateness of the utility function'.

Lange (1904–1965) was a Polish economist who studied at the University of Cracow, where in 1927 he became a lecturer in statistics. In 1929 he visited London and Cambridge. Lange was another early member of the Econometric Society, participating in its Paris (October 1932) and Leyden (September–October 1933) meetings. These were small gatherings in which the participants (22 in Paris and 30 in Leyden; Lutfalla 1933, Marschak 1934) became acquainted. At Leyden, with Allen, Hicks, Lerner and Webb also in attendance, Lange presented a paper (Lange 1932) that was approvingly referred to by Hicks and Allen (1934: 64). In 1934, Lange left Poland on a two-year Rockefeller fellowship that brought him first to the LSE and then to Harvard, where he studied under Joseph Schumpeter. In 1938, he returned to the United States, and the following year was appointed professor at Chicago (Dobb 1966).

In his article on the determinateness of the utility function, Lange summarized the discussion on the implications of the immeasurability of utility from Pareto to Hicks and Allen. He labeled the assumption that individuals are only able to rank combinations of goods 'postulate 1', and the assumption that individuals are also capable of ranking transitions from one combination of goods to another 'postulate 2'. For Lange, none of the economists who admitted postulate 2, namely Pareto, Amoroso, Bowley, Rosenstein-Rodan and Morgenstern, seemed to have realized that it implies something that they discarded as implausible, namely that individuals are also capable of stating how many times a given transition is preferable to another (this is Pareto's point 3):

> From the assumption that the individual is able to know *whether* one increase of utility is greater than another increase of utility the possibility of saying *how many times* this increase is greater than another one follows necessarily. (Lange 1934a: 220)

In fact, Lange reasoned, if postulate 2 holds we can vary combination III until the individual perceives the change of utility due to transition from II to III as equally preferable to the change of utility due to transition from combination I to combination II, that is, we can vary combination III until U(III)–U(II)=U(II)–U(I). Rearranging this equation, we obtain U(III)–U(I)=2[U(II)–U(I)], and thus that 'the change of utility due to transition from I to III is *twice* the change of utility due to transition from I to II' (222). Therefore, Lange concluded, postulates 1 and 2 imply a return to 'determinate', i.e. measurable, utility:

> The two fundamental assumptions used by Pareto and other writers of his and of the Austrian school (and by Professor Bowley) are equivalent to the assumption that utility is measurable. (223)

In making this point, Lange also ushered onto the stage the positive linear transformations of the utility function that are the protagonist of our narrative. As Pareto had shown in the *Manual*, postulate 1 restricts the admissible transformations of U to the positive ones, that is, to $F(U)$ having a positive first derivative: $F'>0$. Postulate 2, argued Lange, further restricts the admissible transformations of U by implying that the second derivative of $F(U)$ is equal to zero: $F''=0$. But the transformations F that display a positive first derivative and a null second derivative are the positive linear ones, that is those of the form $F(U)=\alpha U+\beta$, where $\alpha>0$ (221).

Based on the conviction that postulate 2 restores the determinateness or measurability of the utility function, Lange indicated two alternative approaches to demand analysis. The first,

based on postulate 1 alone, is sufficient to establish all equations of demand analysis. This approach reduces the assumptions to one, and this single assumption 'can be expressed in terms of objective human *behaviour*, i.e. in terms of choice' (224). The second is based on postulates 1 and 2. Postulate 2 cannot be expressed in terms of choice behavior and, to have some insight about which transition an individual prefers, we have to rely on the individual's communication of the result of psychological introspection. For Lange, the main merits of the second approach are that it allows for a psychological interpretation of the equations of demand theory in terms of intuitive concepts such as decreasing marginal utility and that it permits a welfare analysis of economic equilibrium. For Lange, both approaches are legitimate.

Some comments on Lange's article are in order. To begin with, while the first to connect explicitly and formally the ranking of utility differences with positive linear transformations of the utility function, he did not employ the expression 'cardinal utility'. Second, Lange made explicit that the technical issue concerning the meaning and implications of the utility-difference ranking is intertwined with the more general methodological issue concerning the relationship between economics and psychology. In particular, he was neither a behaviorist nor a stern ordinalist, but acknowledged the fruitfulness of psychological introspection in economics. Finally, like Pareto, Amoroso and the other economists who admitted postulate 2, Lange took for granted that the ranking of transitions from one combination to another and the ranking of utility differences are one and the same thing. Phelps Brown's comment on Lange's article showed that this is not the case.

6. The analogy of quantity and Phelps Brown's critique

Like Hicks, Henry Phelps Brown (1906–1994) studied at Oxford, where he was taught by Robbins, and whom he replaced as lecturer in economics at New College when the latter moved to the LSE. He spent 1931–32 as a Rockefeller Traveling Fellow in the United States, visiting various universities including Chicago, where he studied under Henry Schultz, a statistician, mathematical economist, admirer of Pareto and founding member of the Econometric Society. Back in Oxford, Phelps Brown focused on statistics and mathematical economics and became an active member of the Society (Hancock and Isaac 1998).

In the Society's Paris meeting of October 1932, Phelps Brown read a paper which argued that willingness to pay cannot be used as a measure of marginal utility (Phelps Brown 1934a). At the Leyden meeting he renewed acquaintanceship with Lange and met also Allen, Hicks, Lerner and Webb from London and Schultz from Chicago. Given his research interests and his acquaintance with Lange, the fact that Phelps Brown commented on Lange's paper on the determinateness of utility is not surprising.

In his three-and-half-page-long comment, Phelps Brown (1934b) showed that the implications of postulate 2 are much weaker than those supposed by all its supporters from Pareto to Lange, and that, in particular, Lange's claim that postulate 2 restricts the admissible transformations of the utility function to the positive linear ones is unwarranted.

Phelps Brown begins by noticing that both postulates 1 and 2 concern preference order. Postulate 1 refers to preference order over combinations of goods and allows for the introduction of a numerical index U that assigns larger numbers to more preferred combinations. Postulate 2 refers to the preference order over transitions from one combination to another, and allows for the introduction of another index, let us call it G, that assigns larger

numbers to more preferred transitions. However, Phelps Brown stressed, the numbers associated by G to transitions need not be equal to the differences between the numbers associated by U to combinations (67).

Thus, if an individual prefers combination III to combination II, and combination II to combination I, then postulate 1 implies that U(III)>U(II)>U(I). If, in addition, the individual prefers transition from I to II to transition from II to III, then postulate 2 implies that G(I, II)>G(II, III). But postulate 2 does not imply that U(II)–U(I)=G(I, II), nor that U(III)–U(II)=G(II, III).[10] Moreover, since postulate 2 refers only to the ranking of transitions and G-numbers and has no implications on the differences between the U-numbers, then Lange's proof that postulate 2 restricts the admissible transformations of utility function *U* to the positive linear ones cannot be correct.

Phelps Brown (68) pointed out a second problem, strictly connected to the previous one. If postulate 2 has no implication on the U-numbers, it also cannot have implications on the variation of marginal utilities as expressed by the differences between those numbers. Therefore, postulate 2 does not allow us to talk meaningfully of decreasing marginal utility or to employ the traditional definition of complementarity.[11]

Finally, Phelps Brown called attention to a third problem: since the G-numbers have only an ordinal meaning, it does not make sense to sum them.[12] Thus, for instance, if the individual considers transition from I to II equally preferable to transition from II to III, then the G-number associated with both transitions is the same, say 7. What will be the G-number associated with the transition from I to III? Since the transition from I to III is obtained by making two equally preferable transitions associated with the G-number 7, it is tempting to answer 7+7=14. But this temptation, argued Phelps Brown, is misleading, and depends on the fact that, in representing preferences by numbers, we tend illegitimately to extend the additive properties of numbers to preferences. If we avoid numbers and, for instance, represent preference orders by the order of words, the temptation to sum what cannot be summed disappears:

> The two included transitions [from I to II, and from II to III] are indistinguishable, and to each will therefore correspond the same term […], *maison*. We have then no temptation to suppose that if the consumer makes the transition represented by *maison* once and then once again, he has made in all a transition to be represented by 2(*maison*). (68)

Therefore, Lange's claim that postulate 2 allows us to say that the change of utility due to a transition is twice the change of utility due to another transition is unwarranted, as his conclusion is that postulate 2 implies a return to determinate or measurable utility.

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[10] Consider the following numerical example. If the individual prefers III to II, and II to I, we can assign the following U-numbers to the three combinations: U(III)=10, U(II)=3, and U(I)=1. If the individual prefers transition from I to II to transition from II to III, we can assign to the two transitions the G-numbers G(I, II)=5 and G(II, III)=2. Although these U-numbers and G-numbers are perfectly consistent with postulates 1 and 2, it turns out that U(II)–U(I)=2 while G(I, II)=5, and U(III)–U(II)=7 while G(II, III)=2.

[11] In our numerical example, where U(III)=10 and U(I)=1, it can be that U(II)=3 or U(II)=8 without violating postulate 2. In the first case, marginal utility is increasing, since U(II)–U(I)=2, while U(III)–U(II)=7. In the second case, instead, marginal utility is decreasing: U(II)–U(I)=7, and U(III)–U(II)=2.

[12] The same holds for the summation of the U-numbers, but Phelps Brown did not discuss this point as it was not relevant to his argument.

To sum up, Phelps Brown showed that the power of postulate 2 is much more limited than had been previously supposed, and that all the nice implications that Pareto, Lange and others had imagined that they had drawn from it had in fact been illusory. The illusion is caused by 'the analogy of quantity' (68), that is, by representing psychological phenomena like preferences through numbers, without keeping in mind that not all properties of numbers extend to preferences:

> [The analogy of quantity], though permissible, is dangerous, because quantities have properties which we cannot easily banish from our thoughts, and some of these properties have no part in the just analogy. It is by the unnoticed intrusion of such properties that the semblance of measurable utility has appeared. (69)

Phelps Brown's article was followed by a note by Lange (1934b) in which he refined his proof that the comparability of differences between the U-numbers restricts the admissible transformations of U to those of the form $F(U)=\alpha U+\beta$, but did not address Phelps Brown's point that postulate 2 does not warrant the comparability of the differences between the U-numbers. It appears that Lange wrote his note before reading Phelps Brown's comment. Nevertheless, in a letter sent to Samuelson on 10 May 1938, Lange acknowledged that Phelps Brown's objections were correct (on this letter, see Section 9.3).

Two brief final comments on Phelps Brown's contribution are in order. First, Phelps Brown did not use the cardinal-ordinal terminology. Second, he did not investigate what assumptions, if any, should be added to postulates 1 and 2 to make sense of the sum of the G-numbers or warrant that the G-numbers coincide with the differences between U-numbers. Thus, Phelps Brown opened, but did not then close an analytical Pandora's Box.

7. Reactions at LSE: Bernardelli and Allen

Lange's article and Phelps Brown's comment prompted the reactions of two affiliates of the Robbin's group, namely Harro Bernardelli and Lange. Despite their common institutional affiliation, Bernardelli and Lange embodied two opposite views regarding postulate 2 and, more generally, the relationship between psychology and economics, as well as the proper approach to demand analysis.

7.1. Bernardelli's defense of postulate 2

A Viennese of Italian extraction, Bernardelli (1906–1981) had studied economics in Bonn and Frankfurt before moving to the LSE in 1933, where he entered the Robbins circle. Following a research fellowship at Liverpool, in 1937 he moved eastward, to universities in, first, Burma (Rangoon) and then New Zealand (Otago) (Donoghue 2007).

In his comment on Lange's article, which was published just after Phelps Brown's note, Bernardelli (1934) accepted without reservation Lange's claim that postulate 2 implies the measurability of utility, and defended the psychological plausibility and scientific legitimacy of the postulate. Opposing both a strict ordinalism admitting only postulate 1 and Hicks's and Allen's behavioristic approach, Bernardelli argued that postulate 2 should be retained as a fundamental pillar of economic analysis, its rejection entailing 'the relinquishing of many propositions which until now have been considered as undoubtedly belonging to the body of Economic Theory', such as the principle of decreasing marginal utility and the traditional and intuitive definition of complementarity (71). For Bernardelli, the theories of Pareto and

Hicks–Allen are 'axiomatic experiments' showing how much of our economic knowledge is independent of the second postulate, and they resemble 'the behaviour of a man who cuts off one of his legs, in order to see how he gets on as a cripple' (71). But such amputation is not necessary:

> It is extraordinary how one can get on without the leg of the second postulate, as the results of Pareto, and more recently of Allen and Hicks, prove. Yet this would seem insufficient reason for making a virtue of such an amputation. (71–2)

Notably, in his comment, Bernardelli did not employ the expression 'cardinal utility'.

7.2. Allen's criticism of postulate 2

With a brief note in the February 1935 issue of the *Review*, Allen also entered the fray, denying the usefulness of postulate 2. Allen argued that, since the theory of value can be developed on the basis of postulate 1 alone and postulate 2 'cannot be expressed in terms of the individual's acts of choice', it would be futile to complicate the analysis with postulate 2 unless it 'works its passage' (1935: 155–6). Notably, Allen took into account Phelps Brown's criticism of Lange but circumvented the difficulties it raised by re-interpreting postulate 2 as concerning directly the capacity of ranking utility increments, that is, of stating whether U(II)–U(I) is larger, smaller or equal to U(III)–U(II), rather than the capacity of ranking transitions as it was in its original formulation.

In opposition to Lange and Bernardelli, who had argued that postulate 2 is necessary to understand complementarity and for welfare analysis, Allen claimed that this was not the case. The new definition of complementarity he and Hicks had proposed in their 1934 article was not only independent of postulate 2, but showed that the distinction between complementary and substitute goods 'has nothing to do with utility or intensities of preference' and is rather based on 'the inter-relations of individual demands under market conditions' (158). Postulate 2 does not even warrant welfare analysis, for which 'additional, and far more serious, assumptions about the relations between the preference scales of *different* individuals are necessary' (158). In conclusion, for Allen postulate 2 does not work its passage and should be discarded.

Like Lange, Phelps Brown and Bernardelli, in his comment Allen did not use the expression 'cardinal utility'.

7.3. Some speculations on Robbins

In the mid-1930s, the leader of the group to which both Bernardelli and Lange were affiliated did not take an explicit stance on postulate 2 or the debate it generated. However, it seems likely that Robbins's views were closer to Bernardelli's than to Allen's.

At a general methodological level Robbins, like Bernardelli, was opposed to behaviorist approaches in economics and defended reference to psychological variables in the explanation of economic phenomena not only as scientifically legitimate, but as scientifically necessary (see e.g. Robbins 1935: 86–8). Based on this methodological stance, Robbins may well have considered postulate 2 as acceptable, at least in principle. Certainly, in an article dealing with utility theory that he wrote almost twenty years later, Robbins (1953) explicitly advocated that individuals are capable of ranking transitions among different combinations of goods:

> I am quite sure that I can and do judge differences. The proposition that my preference for the Rembrandt over the Holbein is less than my preference for the Holbein over, let us say, a Munnings, is perfectly intelligible to me. (104)

This statement may reflect views Robbins arrived at only after the mid-1930s, but suggest that even in that period he could have considered postulate 2 as scientifically legitimate and psychologically plausible.

8. The man who came in from mathematics: Alt's 1936 contribution

We have observed that Phelps Brown did not investigate what additional assumptions could warrant the identification of the ranking of transitions from one combination to another with the ranking of utility differences. We have also mentioned that Bernardelli apparently did not notice the problem, while Allen begged the question by re-interpreting postulate 2 as directly concerned with the capacity of ranking utility differences. The man who closed the Pandora's Box opened by Phelps Brown was Franz Alt, a young Viennese mathematician and economist little known in the history of economics.

8.1. A biographical sketch

Alt (1910–2011) graduated in mathematics from Vienna in 1932 with a dissertation under Karl Menger, the son of the founder of the Austrian school of economics. Menger's mathematical research concerned the theories of curves, dimensions, and metric spaces, and in his dissertation Alt provided a definition of the curvature of a curve that generalized a definition previously given by Menger himself. From 1930 Alt became a habitual participant in Menger's seminar, the *Mathematische Kolloquium*, whose importance in the development of economics, particularly in relation to general equilibrium theory, has been extensively investigated by historians of economics.[13] As a Jew, Alt failed to obtain an academic position, but on Menger's recommendation was appointed by Morgenstern as a private tutor in mathematics.[14]

Thanks in no small measure to his experiences abroad in the period 1925–28, by the late 1920s Morgenstern had distanced himself from the typically Austrian distrust for the application of mathematics to economics, and had become increasingly interested in mathematics, logic and their application to the social sciences. From around 1933 Morgenstern's attraction to mathematics was accelerated by his intense engagement with Menger (Leonard 2010), and when the former decided to improve his mathematical skills, the latter persuaded him to hire his student Alt. Morgenstern and Alt met for a couple of hours a week, read together books in mathematical economics, such as Bowley's *Mathematical Groundwork of Economics* (1924), and discussed each chapter in detail (Alt and Akera 2006: 7).

Through participation in the *Kolloquium* and his tutoring of Morgenstern, Alt became interested in the mathematical aspects of economics, as testified by two articles (Alt 1935, [1936] 1971) and a number of reviews of economics books published in Morgenstern's *Zeitschrift für Nationalökonomie* between 1934 and 1938.

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[13] See, among others, Weintraub (1983), Punzo (1989), Ingrao and Israel (1990).

[14] Menger found a similar solution for another bright Jewish student, namely Abraham Wald, who became the private tutor in mathematics of the banker and economist Karl Schlesinger. On Wald, see Weintraub (1983) and Leonard (2010).

Through Morgenstern, at some point between late 1934 and early 1935 Alt became involved in the debate over the determinateness of the utility function:

> I remember […] an afternoon tea at Morgenstern's house to which I was invited. I met a whole lot of people there, among others, an American called [Paul] Sweezy, a mathematical economist. […] Paul Sweezy showed me a reprint that he was carrying from a Polish mathematician, […] Oskar Lange, about measuring the value of economic commodities. (Alt and Akera 2006: 7)

Alt had been trained in the axiomatic mathematical tradition of David Hilbert, which was also the standard approach of Menger and other *Kolloquium* participants. In this tradition, one specifies a consistent set of axioms and proves that a given statement of interest can be logically deduced from them.[15] With respect to the standards of proof accepted in mathematics, Alt found Lange's demonstration that postulates 1 and 2 imply the measurability of utility unsatisfactory, and began writing a letter of reply to Lange that ended up becoming his 1936 article:

> Lange said if you had these two conditions [postulates 1 and 2] then that's sufficient to assign a number to every commodity by itself. […] I read that, and I was a very theoretical mathematician. That's not mathematics. That's not a proof, I thought. I began to write a letter to Oskar Lange […], and the letter grew to be 10 pages long. And I realized I was writing a paper. (8–9)

Originally written in English, Alt translated his paper into German and gave it to Morgenstern. It was published as 'Über die Messbarkeit des Nutzens' (On the measurability of utility) in the June 1936 issue of the *Zeitschrift*.[16]

The story of Alt's article has an American sequel. In May 1938, a few weeks after the annexation of Austria by Nazi Germany, Alt and his wife fled to New York. At that time his two Austrian mentors were already in United States: Menger had obtained a position at the University of Notre Dame, Indiana, while Morgenstern had joined Princeton University. They helped Alt find a job by writing recommendation letters on his behalf.[17] After some time he was hired as an econometrician at the Institute of Applied Econometrics, New York. Around 1939, Alt received a letter from another Austrian *émigré*, namely Schumpeter. Schumpeter wrote that he had read the *Zeitschrift* article, and asked Alt whether he was still working on the measurability of utility. Schumpeter was possibly fishing to see whether Alt would be interested in a job at Harvard, but Alt did not catch his intention in time (Alt and Akera 2006: 9). In his *History of economic analysis*, Schumpeter (1954: 1063) acknowledged that Alt's 1936 article provided a satisfactory solution to the issue of utility measurement.

To end our biographical sketch of Alt, we mention that in 1943 he volunteered for the American Army, and after the war he became a leading scholar in the rising field of computer science (Alt 2001, Alt and Akera 2006).

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[15] On the different aspects of Hilbert's axiomatic method, see Weintraub (2002).

[16] In July 1936 Alt presented a two-and-half-page-long English version of the paper at the International Congress of Mathematicians held in Oslo, in which he participated together with Menger. This short English version was published the following year in the proceedings of the Congress (Alt 1937). A one-sentence English abstract of Alt's article also appeared in the section 'Recent periodicals and new books' of the September 1936 issue of the *Economic Journal* (Anonymous 1936: 574).

[17] See Morgenstern's letter to Alt of 20 July 1938, and Alt's reply of 25 July 1938. Both letters can be found in Morgenstern papers, box 4, folder 'Correspondence Series, 1918–1977; 1928–1939: A–El'.

8.2. The paper's content

As mentioned, Alt's 1936 contribution to the determinateness-of-the-utility-function-debate was motivated by his criticism of Lange's supposed demonstration that postulates 1 and 2 imply the measurability of utility. In particular, Alt agreed with Phelps Brown that the key flaws of Lange's argument consisted in the unwarranted intermingling of the preference order over combinations with the preference order over transitions, and in the tacit attribution of additive properties to the latter (Alt [1936] 1971: 431).

In the spirit of the axiomatic method, Alt added to Lange's two postulates five additional postulates concerning the properties of the two preference orders and their relationships. In particular, Postulates 3 and 6 require that both preference orders are transitive and continuous.[18] Postulate 4 and Postulate 7 connect the two preference orders.[19] Postulate 5 provides the preference order over transitions with an additive structure.[20]

Alt proved that these seven postulates are necessary and sufficient for the existence of utility function U over combinations of goods such that: (i) combination x is preferred to combination y if and only if $U(x)>U(y)$; (ii) the transition from y to x is preferred to the transition from w to z if and only if $U(x)–U(y)$ is larger than $U(z)–U(w)$; and (iii) U is unique up to positive linear transformations. Alt thus provided an analytically rigorous answer to the question concerning the exact conditions that make utility measurable in the sense envisaged by Lange.

Alt also addressed the validity and empirical verifiability of the seven postulates. He believed that postulate 1 'can be verified *by economic observations*' and is therefore well-founded (431). In opposition to Bernardelli, Alt found postulate 2 more problematic because it is not clear 'whether it is at all possible to make comparisons between transitions […] on the basis of experience' (431). With respect to the other five postulates, Alt left the issue concerning their validity open, arguing that they 'can (and must) be tested against experience' (431).

8.3. Comments on Alt's article

Three final comments on Alt's article are in order. First, Alt's proof concerning the conditions delivering utility unique up to positive linear transformations differs from Lange's not only because the former is mathematically correct while the latter is not, and because Alt followed Hilbert's axiomatic approach while Lange did not. Also, the kind of mathematics they used is

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[18] A preference relation is continuous if it is preserved under limits. In the 1930s, requiring the continuity of the preference relation was quite exceptional. The importance of this assumption came to be appreciated only in the mid-1950s, when Nicholas Georgescu-Roegen's paradox of lexicographic preferences showed that transitivity of preferences alone in general does not warrant the existence of a utility function representing them, and that continuity is also required. More on this in Moscati (2007).

[19] Postulate 4 states that the individual prefers combination x to combination y if and only if he prefers the transition to x to the transition to y whatever the starting combination z is and, at the same time, he prefers reaching whatever combination w by starting from y rather than by starting from x. Postulate 7 is an Archimedean requirement: if x is preferred to y, there exists a finite sequence of equivalent transitions to more preferred combinations such that the last element of the sequence is at least as preferred as x.

[20] Postulate 5 requires that, if transition from x to y is preferred to transition from x' to y', and transition from y to z is preferred to transition from y' to z', then transition from x to z is preferred to transition from x' to z'.

different. While Lange employed the differential calculus, that is, the kind of mathematics traditionally used in economics since the marginal revolution, Alt used topological analysis, which at the time was a thriving field of mathematical research. As with the axiomatic approach, topological techniques were also quite exceptional in economics in the 1930s, becoming more common in economic theory only in the 1950s.

Second, the fact that Alt's article was published in German in an Austrian journal seems to have hindered its appreciation in the Anglo-Saxon academic world. We have seen that it was known by a number of important Austrians who emigrated to the United States, namely, Menger, Morgenstern, and Schumpeter, and in the next Section we will discover that also Lange, who was Polish but fluent in German, read Alt's paper. However, in the journals and books collected in the JSTOR database, in the ten years after its publication Alt's article was cited only twice and even then only in footnotes: by George Stigler (1938: 575, footnote 8) and Gerhard Tintner (1942: 275, footnote 8).[21]

Finally, Alt did not use the expression 'cardinal utility' to label utility that is unique up to positive linear transformations. One may speculate that, even if Alt was aware of the cardinal-ordinal terminology, as a mathematician he associated the term cardinal with one of the two main meanings it had in mathematics. In a first meaning, as mentioned in Section 4, cardinal numbers express the total number of units constituting a given quantity. The second meaning is associated with the theory of transfinite sets put forward by Georg Cantor ([1887] 1932). Here, cardinal numbers characterize a family of sets whose elements can be put into a one-to-one correspondence. However, in neither mathematical meaning did cardinal numbers have anything to do with the ranking of differences between objects or uniqueness to positive linear transformations.

In fact, it was the economist Paul Samuelson who, in 1938, coupled 'cardinal utility' with utility unique up to linear transformations.

9. Cardinal utility in Samuelson's early work

9.1. Utility, discounted utility, and choices

Samuelson (1915–2009) entered the University of Chicago in 1932 and then moved on to Harvard Graduate School, where he studied under, among others, Joseph Schumpeter. In 1937 and aged only twenty-one, Samuelson published in the *Review* his first scientific article, 'A Note on Measurement of Utility'.[22] Here, he put forward a model of intertemporal choice where the individual behaves so as to maximize the discounted sum of all future utilities. This discounted-utility model quickly became the standard neoclassical formalization of intertemporal choice. For our concerns, it is important to notice that for Samuelson the maximization of the discounted sum of future utilities implies that the individual is able to rank utility differences, i.e., Pareto's postulate 2:

[21] Neither Stigler nor Tintner had problems with the German language: the first was the son of immigrants from Bavaria and Austria-Hungary, while the second was another Austrian economist who emigrated to the United States in the late 1930s.

[22] In the June 1937 issue of the *Review*, the Danish economist Frederik Zeuthen published a note that, at least in its title, also related to the determinateness-of-the-utility-function debate (Zeuthen 1937). In this note, Zeuthen criticized the behaviorist approach to demand analysis and argued that introspection is necessary to economic theory. However, Zeuthen did not elaborate on issues concerning cardinal utility.

> Reflection as to the meaning of our Assumption Two [that the individual maximizes the sum of future utilities] will reveal that the individual must make preferences in the Utility dimension itself, that is to say, we must invoke Pareto's Postulate Two, which relates to the possibility of ordering *differences* in utility by the individual. (Samuelson 1937: 160–1)

The above quotation also shows that, following Lange and ignoring the contributions of Phelps Brown and Alt, Samuelson in 1937 identified postulate 2 with the possibility of ranking utility differences. Accordingly, he claimed that postulate 2 restricts the admissible transformations of the utility function to the positive linear ones. However, in his first publication Samuelson did not use the cardinal-ordinal terminology.

As is well-known, the 1937 article was the first of an exceptionally copious and long-lasting series. In 1938 alone, Samuelson published four articles in major economics journals, three of which were related to utility theory and demand analysis, while the fourth addressed welfare economics.

The first 1938 article is Samuelson's celebrated 'Note on the Pure Theory of Consumer's Behaviour' (1938a). Here, the Harvard Ph.D. student criticized Hicks and Allen's demand analysis as not properly behaviorist, and put forward his own brand of behaviorism, later called the revealed preference approach to consumer demand (1938a).[23] Since the goal of the 'Note' was to show that demand analysis requires no reference to utility, Samuelson did not dwell on the issues concerning the notion of utility. However, and this is important for the terminological aspect of our story, he employed the expression 'cardinal utility' for the first time in print. In reviewing the history of demand analysis based on utility, Samuelson argued that it had progressively ruled out unnecessarily restrictive conditions such as 'the assumption of the measurability of utility in a cardinal sense' (61). It is not clear, however, what 'measurability of utility in a cardinal sense' means, and the expression is not associated with utility unique up to positive linear transformations.

In the second 1938 article, 'The Empirical Implications of Utility Analysis', Samuelson (1938b) argued that the ordinal utility theory initiated by Pareto does have refutable implications in terms of demand behavior, such as the negativity of the substitution effect, and attempted to provide a complete list of these implications. However, Samuelson claimed, the same implications can be derived more easily and directly from the postulates on choices he had put forward in the 'Note'. In this article Samuelson twice employed the expression 'ordinal preference' (345), but not the term 'cardinal utility'.

9.2. Samuelson's cardinal utility

Samuelson's third article of 1938, 'The Numerical Representation of Ordered Classifications and the Concept of Utility', appeared in the October 1938 issue of the *Review* (1938c) and is the most relevant one for our story. Samuelson here provided his solution to the problem concerning the conditions restricting the admissible transformations of the utility function to the positive linear ones, and consistently coupled the expression 'cardinal utility' with utility unique up to those transformations.

At the outset of the article, Samuelson acknowledged that Phelps Brown was right in criticizing Lange's results because they were based on an unwarranted identification of the G-numbers representing the ranking of transitions with the difference between the U-numbers

[23] The analysis of Hicks and Allen relied on the assumption that the marginal rate of substitution is decreasing, i.e., that the indifference curves are convex. However, for Samuelson (1938a: 61) this assumption depends on introspection and therefore is not sound; see Moscati (2007).

representing the ranking of combinations (65). Now, Samuelson saw that this identification cannot be taken for granted and accordingly investigated under what conditions it is valid.

In effect, the issue concerning the hypotheses that warrant the identification of the G-numbers with the difference between the U-numbers is exactly the problem that Alt had already addressed and solved in his 1936 article. However, Samuelson did not mention Alt's article.

Following Phelps Brown, Samuelson began by noticing that postulates 1 and 2 concern only preference order, that Postulate 1 allows for the introduction of an index U that assigns larger numbers to more preferred combinations, and that Postulate 2 allows for the introduction of another index G assigning larger numbers to more preferred transitions (65–8). Then Samuelson assumed that both preference orders are transitive, and informally connected them by arguing that if an individual prefers the transition from x to y to the transitions from x to z, that is, if $G(x, y)>G(x, z)$, then combination x must be preferred to combination z, that is, $U(y)>U(z)$. This informal assumption corresponds to Alt's Postulate 4.

Subsequently, Samuelson introduced the key postulate of his article as equation 15 (68). We have mentioned in Section 6 that Phelps Brown had also showed that Postulate 2 does not warrant the possibility of summing G-numbers. Samuelson's postulate overcomes the problem by simply assuming that G-numbers can indeed be summed. That is, if $G(x, y)$ is the number associated with the transition from x to y, and $G(y, z)$ is the number associated with the transition from y to z, the postulate requires that the number $G(x, z)$ associated with the transition from x to z, must be equal to the sum of $G(x, y)$ and $G(y, z)$, i.e., $G(x, y)+G(y, z)=G(x, z)$. This postulate corresponds to Alt's postulates 5 and 7. But while Alt's postulates concerned the preference orders over combinations and transitions, Samuelson's assumption refers directly to the G-numbers and therefore does not make clear what features of the preference orders may be behind it.

At any rate, Samuelson showed that this postulate, together with the other assumptions mentioned above, is necessary and sufficient to make the G-numbers associated with transitions equal to the difference between the U-numbers associated with combinations, that is, to have $G(x, y)=U(y)–U(x)$. In turn, as Lange had already showed, $G(x, y)=U(y)–U(x)$ if and only if the utility function U is unique only up to linearly increasing transformations (69–70).

In the final part of his paper, Samuelson discussed the plausibility of the condition $G(x, y)+G(y, z)=G(x, z)$, and argued that it is an 'arbitrary restriction' that must be regarded as 'infinitely improbable' (70). Therefore, he concluded, the uniqueness of the utility function up to positive linear transformations should also be considered as arbitrary and infinitely improbable.

Thus, Samuelson's confidence in the plausibility of cardinal utility shifted from the agnosticism of his discounted-utility article of 1937 (where, in effect, cardinal utility was necessary to make sense of the discounted-utility model) to the disbelief expressed in the 1938 paper under examination.

This last work also contains a terminological novelty that is central to our story: for the first time utility unique up to positive linear transformations was explicitly and consistently coupled with the terms 'cardinal' and 'cardinal measurability'. This association occurs ten times in Samuelson's paper, of which one is selected as an example:[24]

[24] The other nine occurrences can be found at pages 65, 68 and 70 of Samuelson (1938c).

> Dr. Lange has not proved satisfactorily that from these two assumptions [Pareto's postulates 1 and 2] can be derived the cardinal measurability of utility (subject to a linear transformation involving scale [the number α in the formula $\alpha U(x)+\beta$] and origin [the number β in $\alpha U(x)+\beta$] constants). (66)

We argue, therefore, that 'cardinal utility' acquired its current technical meaning in Samuelson's 1938 article.

9.3. Samuelson, Lange and Alt

One question that naturally arises is whether Samuelson knew of Alt's 1936 article. We can say that, at the least, he was aware of its existence.

Presumably in early 1938, Samuelson sent a draft of his cardinal-utility paper to Lange, who replied in the letter of 10 May 1938, mentioned above.[25] Lange declared Samuelson's manuscript 'a contribution which really helps to clarify the subject', and judged Samuelson's equation 15, that is his postulate $G(x, y)+G(y, z)=G(x, z)$, as a satisfactory solution to the problems Phelps Brown had called attention to:

> I agree with your argument and particularly that the functional equation (15) is necessary to establish measurability. It was contained implicitly in my formulation of postulate (2) […]. It was exactly (15) that Phelps Brown had in mind when he objected to my argument. The formulation of the postulate 2 given by me was simply that of Pareto, Bowley, etc. since I was chiefly concerned with the inconsistency of their argument.

But in his letter, Lange also explicitly invited Samuelson to look at Alt's article, and pointed out the possible relationship between Samuelson's postulate 15 and Alt's postulates

> I would suggest that you look up the article of Alt, Über die Messbarkeit des Nutzens, Zeitschr. F. Nat.- Oeconomie, Bd. VII (1936). If I am not mistaken your equation (15) corresponds to his postulates IV and V.

We know from a letter of Ursula Webb Hicks to Samuelson that he did not see the proofs of his article.[26] Therefore, even if Samuelson looked at Alt's article between May and October 1938, he could not add a reference to Alt. It is noteworthy, however, that in his subsequent writings of the 1930s and 1940s Samuelson did not refer to Alt's 1936 article.[27]

In his *History of economic analysis*, Schumpeter touched upon Samuelson's possible acquaintance with Alt's article. As we have seen (Section 8), Schumpeter appreciated Alt's article; but he was also one of Samuelson's mentors at Harvard and followed his pupil's research on demand analysis. So, it does not seem implausible that, at some point, Schumpeter mentioned Alt's paper to Samuelson. At any rate, in his *History*, after having pointed to Alt's 1936 article as providing a satisfactory solution to the issue of utility measurement, Schumpeter (1954: 1063) turned to Samuelson's (1938c) paper and declared that 'Alt's argument […] was not known to Samuelson'.

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[25] The letter can be found in the Samuelson papers, box 48, folder 'Lange, Oskar, 1938–1946'.

[26] Letter of Ursula Webb Hicks to Samuelson, 4 October 1938, Samuelson papers, box 37, folder 'Hicks, Mrs. Ursula, 1949–1960'.

[27] Samuelson's first reference to Alt's article is contained in a footnote of Samuelson (1950).

9.4. Utility, or not utility?

A final discussion concerning Samuelson is in order. According to a widespread image, Samuelson's chief goal during the ordinal revolution was to free economic theory from any vestige of the utility concept (see e.g. Wong 2006). However, the four articles examined in this section (Samuelson 1937, 1938a, 1938b, 1938c) show that this conventional account is inadequate. Eliminating the utility concept from economic theory may have been the goal of the 'Note' (1938b), but it was certainly not the goal of the other three papers. In them, Samuelson dealt intensively with utility analysis and contributed substantially to it, especially through his discounted-utility model for intertemporal choice of 1937.

Samuelson also employed a utilitarian approach in his 1938 article on welfare analysis (1938d), as well as in a subsequent polemical exchange with Bernardelli (1939). And even in his Harvard Ph.D. dissertation, which was delivered in November 1940 and that became, with few modifications, the *Foundations of economic analysis* (1947), Samuelson (1940) played down the revealed-preference approach proposed in the 'Note' and presented the theory of consumer demand more or less according to the ordinal-utility approach.

We may speculate about the reasons for Samuelson's mixed attitude toward utility between 1937 and 1940. On the one hand, at that time he was still a Ph.D. student in his early twenties, and possibly he was still exploring different research paths. A certain scientific opportunism may also have played a role in his swinging between utilitarianism and behaviorism: he adopted a utilitarian approach, possibly based on cardinal utility, when this appeared necessary to obtain interesting theoretical outcomes, as was the case in the article on intertemporal choice; however, when a certain theoretical result could be obtained from postulates on choices, as in the case of the 'Note', Samuelson was glad to leave utility aside.

Unfortunately, in the part of Samuelson's papers that I examined in his archives at Duke, I did not find any unpublished material that sheds new light on his 'reluctant utilitarianism' during the late 1930s. However, and this is the only point I would like to make here, even a quick considerations of what he published in that period makes clear that the conventional portrait of the young Samuelson as a knightly Saint George relentlessly fighting the utilitarian Dragon is misleading.

10. Von Neumann and Morgenstern's restatement of cardinal utility

In the period between 1938 and 1944, cardinal utility in the specific sense established by Samuelson, i.e., as utility unique up to positive linear transformations, remained peripheral in economics. In effect, it was at odds with the ordinal approach that, especially after the publication of Hicks's *Value and capital* (1939), dominated demand analysis. Not surprisingly, in that book Hicks did not discuss utility unique up to positive linear transformations, did not refer to postulate 2, did not mention the discussion on the determinateness of the utility function initiated by Lange, and did not even use the expression 'cardinal utility' that he had employed in the article co-authored with Allen in 1934.

In works published between 1938 and 1944, Samuelson himself (1939, 1942) referred a couple of times to cardinal utility, but maintained a skeptical stance on its empirical validity and theoretical usefulness.[28] Other economists, meanwhile, began to refer to cardinal utility in

[28] In his Ph.D. dissertation, Samuelson discussed cardinal utility critically but at length, and investigated its connections with other special assumptions of utility theory, such as the

the sense established by Samuelson. Frank Knight, one of the leaders of the Chicago School, referred approvingly to cardinal utility in two articles where he criticized the ordinal approach to demand analysis epitomized by Hicks's *Value and capital* (Knight 1940, 1944). In 1943, Robert L. Bishop, a young colleague of Samuelson at the Massachusetts Institute of Technology,[29] published the first economics article containing the expression cardinal utility in its title. Bishop (1943) argued that cardinal utility is necessary to make sense of the notion of consumer surplus, which he considered useful for welfare analysis.

In 1944 von Neumann and Morgenstern published their *Theory of Games and Economic Behavior*, which among other things, put forward Expected Utility Theory. As already mentioned in Section 1, the axioms of this theory concern the individual's preferences over risky alternatives and imply the existence of a utility function that is unique up to positive linear transformations. Although von Neumann and Morgenstern called this specific form of utility 'numerical utility' rather than cardinal utility, they were perfectly aware that, from Pareto on, it had been connected to the ranking of utility differences:

> [Assume that] there is a criterion with which to compare the preference of C over A with the preference of A over B. It is well known that thereby utilities – or rather differences of utilities – become numerically measurable. That the possibility of comparison between A, B, and C only to this extent is already sufficient for a numerical measurement of 'distances' was first observed in economics by Pareto. (18)

Like Samuelson or Allen, von Neumann and Morgensten were also skeptical about Pareto's postulate 2. However, they showed that the admissible transformations of the utility function can be reduced to the linear ones also in a different way, namely on the basis of a set of apparently less problematic axioms concerning preferences over risky alternatives. Thus, von Neumann and Morgensten saw their approach to 'numerical' or 'cardinal' utility as a safer and simpler alternative to the approach originally suggested by Pareto and developed in the discussion of the 1930s: 'Our procedure, as distinguished from Pareto's, is not open to the objections based on the necessity of artificial assumptions and a loss of simplicity' (29, footnote 4).

The fact that the authors of *Theory of Games* were well aware of the pre-1944 history of cardinal utility should not be surprising. We have seen that in 1931 one of them, namely Morgenstern, had contributed to the debate on the ranking of transitions and had even argued that individuals are capable of comparing utility differences. Moreover, Morgenstern was familiar with Alt and his 1936 article 'On the measurability of utility'. In effect, the passages quoted above and referring to the pre-1944 history of cardinal utility can be found in Chapter 1 of *Theory of Games*, which is the chapter primarily written by Morgenstern (Leonard 1995; Rellstab 1992).

In the years immediately following the publication of *Theory of Games*, the exact meaning and legitimacy of von Neumann and Morgensten's 'procedure' to obtain utility unique up to positive linear transformations was the topic of a heated debate among economists (Ellsberg 1954 provides an early assessment). Also through this debate, this specific type of utility became a fundamental analytical tool of post-World-War-II economic theory.

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independence of marginal utilities (1940: 147–89). This discussion passed with almost no modification into the *Foundations* (1947: 173–202).

[29] Samuelson had left Harvard for MIT in 1940.

These developments are beyond the scope of the present paper, but a final comment is in order. The economists involved in the debate did not call utility unique up to positive linear transformations 'numerical utility' but, rather, consistently referred to it as 'cardinal utility'. This usage definitely cemented the association of 'cardinal utility' with utility invariant to transformations of the form $\alpha U(x)+\beta$ originally established by Samuelson.

12. Conclusions

This paper has reconstructed the discussions that, between 1909 and 1944, progressively led to the definition and stabilization of the notion of cardinal utility in the specific sense it has in current economic theory, that is, as utility unique up to positive linear transformations. Many major economists of the period contributed to these discussions, such as Pareto, Amoroso, Bowley, Rosenstein-Rodan, Morgenstern, Hicks, Allen, Lange, Phelps Brown, Samuelson, Schumpeter and Knight; the less known Franz Alt and Harro Bernardelli also participated in the debate in a significant way.

By illustrating how cardinal utility entered economic analysis in discussions strictly related to the ordinal revolution, the paper has also showed that the two conventional narratives about the origins of cardinal utility – which suggest that cardinal utility entered economic analysis either before or after the ordinal revolution – are flawed.

The history of the consolidation of the notion of cardinal utility includes various dimensions, which the paper has illustrated and connected. The analytical dimension concerns the exact assumptions under which the utility function is unique only up to linear transformations. The epistemological dimension is related to the fact that the debate on cardinal utility illuminated the differences between the properties of preferences and those of numbers, and became a battlefield where the behaviorist and psychologically-based approaches to demand analysis clashed during the ordinal revolution. The paper also investigates the personal and institutional dimensions of the history of cardinal utility, and in particular it sheds light on Alt's figure and his role in the history of utility theory. Finally, the paper explore the possibly unexpected Samuelsonian dimension of the history of cardinal utility, and calls attention to Samuelson's pivotal role in defining and popularizing the current meaning of cardinal utility.

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